DECOMPOSITION OF MATRICES INTO PRODUCT OF IDEMPOTENTS AND SEPARATIVITY OF REGULAR RINGS

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Dedicated to Abraham Berman on his 80^{th} birthday.

Abstract. Following O'Meara's result [Journal of Algebra and Its Applications Vol **13**, No. 8 (2014)], it follows that the block matrix $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$, $B \in M_n(R)$, $r \geq 1$, over a von Neumann regular separative ring R, is a product of idempotent matrices. Furthermore, this decomposition into idempotents of A also holds when B is an invertible matrix and R is a GE ring (defined by Cohn [New mathematical monographs: **3**, Cambridge University Press (2006)]). As a consequence, it follows that if there exists an example of a von Neumann regular ring R over which the matrix $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$ where $B \in M_n(R)$, $r \geq 1$, cannot be expressed as a product of idempotents, then R is not separative, thus providing an answer to an open question whether there exists a von Neumann regular ring which is not separative. The paper concludes with an example of an open question whether every totally nonnegative matrix is a product of nonnegative idempotent matrices.

1. Introduction and notation

Decompositions of singular matrices over a field or division ring (cf. J. Erdos, Laffey) as product of idempotent matrices led to the study of the decomposition of certain elements of a ring as a product of idempotents. There is a plenty of literature related to this question, where some of these are concerned with semigroups, nonegative matrices, or von Neumann regular rings (cf. Gould, Jain-Leroy, O'Meara). All rings are unital. $U(R)$, $E(R)$, and $\Pi E(R)$ denote the set of units, the set of idempotents and the set of elements in R that are product of idempotents, respectively.

We now give some basic facts about the decompositions of zero divisors of a ring into products of idempotents.

If an element a of a ring can be written as product of idempotents then both its left and right annihilators are nonzero. This leads to the study of the following property

$$\forall a \in R,\ l(a) \neq 0 \Leftrightarrow r(a) \neq 0 \qquad (*)$$

We note that unit regular rings, artinian rings, matrix rings over quasi euclidian rings, amongst others, have this property (*) (cf. [10])

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For example, for any ring S, consider the matrix $A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R = M_2(S)$. Clearly $l(A) \neq 0$ and, if A is a product of idempotent matrices, we must have $r(A) \neq 0$. Of course, this condition is not sufficient to write A as a product of idempotents. For instance, if $S = \mathbb{Q}[X, Y]$ and $X = x, Y = y$ we have $XY \in XS \cap YS$. But it is well known that, in this case, the matrix A is not a product of idempotent matrices ([7]).

We collect some basic facts around matrices over any ring with a zero row. These matrices will be the main topic of the next section. For any matrix $A \in M_n(R)$ we write $A_i = (1 - e_{ii})A$, the matrix obtained from A by replacing its i^{th} row by a zero row.

2. Preliminaries

Lemma 1. *(1) Suppose $u \in U(R)$, $e \in E(R)$, and $r \in eR$ are such that $eu, u^{-1}r \in \Pi E(R)$ then $r, ru, ur \in \Pi E(R)$.*

(2) Let $A \in M_n(R)$ with rows $L_1, \ldots, L_n$ be such that $L_i = \Sigma_{j \neq i} \alpha_j L_j$ with $1 - \Sigma_{j \neq i} \alpha_j \in U(R)$. Then if $A_i = (1 - e_{ii})A$ is a product of idempotents, the same is true for A.

(3) Let R be a projective free ring then any matrix $A \in M_n(R)$ which is a product of proper idempotent matrices is similar to a matrix with a zero row (column).

Proof. (1) We have $r = er = euu^{-1}r$. Hence our hypothesis shows that $r \in \Pi E$. We also have $u^{-1}ru \in \Pi E$ and hence $ru = eru = euu^{-1}ru \in \Pi E$. Finally $ur = u(ru)u^{-1}\Pi E$.

(2) Let $u^{-1} = \Sigma_{j \neq i} e_{jj} + (1 - \Sigma_{j \neq i} \alpha_j) e_{ii} \in Gl_n(R)$. We then have have $u^{-1}A = A_i = (1 - e_{ii})A$ and we check that the result follows by applying (1).

(3) If A is a product of idempotent matrices then there exists $E = E^2 \in M_n(R)$ such that $A = EA$. Since R is projective free there exists an invertible matrix $P \in M_n(R)$ such that PEP^{-1} is a diagonal matrix with zero and one on the diagonal. We thus have $PAP^{-1} = PEP^{-1}PAP^{-1}$ and hence PAP^{-1} has a zero row. □

The first statement of Lemma 1 shows that it is important to know when an invertible element $u \in U(R)$ and an idempotent element $e = e^2$ are such that $eu \in \Pi E$. We will give two examples of such a behavior in the frame of matrix rings. Remark first that a matrix in $M_n(R)$ with its i^{th} row zero belongs to $(1 - e_{ii})M_n(R)$. Let us recall that a matrix $A \in M_n(R)$ is a permutation matrix if it each row and each column has only one nonzero entry equal to 1.

Definition 2. *A matrix $A \in M_n(R)$ is a quasi permutation matrix (resp. quasi elementary matrix) if there exists $1 \leq i \leq n$ and a permutation matrix P (resp. elementary matrix Q) such that either $A = (1 - e_{ii})P$ or $A = P(1 - e_{ii})$ (resp $A = (1 - e_{ii})Q$ or $A = Q(1 - e_{ii})$).*

It was proved in [1] that quasi permutation matrices are products of idempotent matrices. Here we prove the similar result for quasi elementary matrices.

Proposition 3. *A quasi elementary matrix is a product of idempotent matrices.*

Proof. Assume that A is a quasi elementary matrix. We note that a quasi elementary matrix always has a zero row and a zero column. Such a matrix is always similar to a matrix having last row zero. In other words, we may assume that our quasi elementary matrix A is of the form $A = (I_n - e_{nn})Q$ where Q is an elementary matrix of the form $Q = I_n + be_{ij}$, where $1 \leq i \neq j \leq n$. If $j = n$, then $A = (I_n - e_{nn})(I_n + be_{in}) = I_n + be_{in} - e_{nn}$. This matrix is easily seen to be idempotent. If $j \neq n$, we have

$$A = \begin{pmatrix} I_{n-1} + be_{ij} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} + be_{ij} & e_i^T \\ -be_j & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

We can easily check that all the three factors are idempotent matrices. □

Proposition 4. *Let $A \in M_n(R)$ be a matrix with a zero row (resp. column) and Q be an elementary matrix or a permutation matrix. Then A is a product of idempotent matrices if and only if QA (resp. AQ) is a product of idempotent matrices.*

Proof. First suppose that the matrix A has its i^{th} row zero and is a product of idempotent matrices. Let $Q = Q_{k,l} = Id + ae_{kl}$ with $a \in R$ be an elementary matrix. If $i \notin \{k, l\}$ then comparing the rows of the matrices on both sides we get that $Q_{k,l}A = (id - e_{ii})Q_{k,l}A$. If $i = k$, we have $Q_{i,l}A = (Id. - e_{ii} + ae_{il})A$ (remark that $Id. - e_{ii} + ae_{il}$ is an idempotent). Finally if $i = l$ we have $Q_{k,i}A = A$. In the three cases we conclude that if A is a product of idempotents then the same is true for $Q_{k,l}A$. Conversely, noting that Q^{-1} is also an elementary matrix, we know, by Proposition 3, that the quasi elementary matrix $(1 - e_{ii})Q^{-1}$ is a product of idempotent matrices. Since the i^{th} row of A is a zero row, we have $A = (Id - e_{ii})A = (1 - e_{ii})Q^{-1}QA$. This shows that if QA is a product of idempotent matrices, the same is true for A.

We leave the proof of the case when Q is a permutation matrix to the reader. □

3. Main results

We now consider a matrix $P \in M_{n+1}(R)$ of the form $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$.

Theorem 5. (1) *If $B \in M_n(R)$ is a product of idempotent matrices then so is $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$, for any column C.*

(2) *If $B \in M_n(R)$ is invertible then $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$ is similar to the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$.*

(3) *If the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is a product of idempotents then so is the matrix $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$ where $C = BQ$ for some column $Q \in M_{n1}(R)$*

Proof. (1) B is a product of idempotent matrices, say $B = E_1 \cdots E_l$ then

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} E_l & 0 \\ 0 & 1 \end{pmatrix}$$

is a product of idempotent matrices.

(2) If B is invertible we can write

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & -B^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & B^{-1}C \\ 0 & 1 \end{pmatrix}$$

(3) If the column C is right linear combination of the columns of B, say $C = BQ$, for some column Q then we have

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & BQ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix}.$$

□

Remark 6. If B is not a product of idempotents (c.f. when B is invertible) it may still happen that the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is a product of idempotents. For example just consider the matrix $B = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \in M_2(k[X, Y])$. The matrix B is not a product of idempotents since the ideal generated by X and Y is not principal. Note $k[X, Y]$ is local and hence projective free, (cf. Lemma 1, [7]). But we can write:

$$\begin{pmatrix} X & Y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We note

$$\begin{pmatrix} X & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & Y \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a product of idempotents.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Thus the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.

Proposition 7. *Let $A, B \in M_n(R)$ be such that $A = CD$ where $C \in M_{n\times r}(R)$ and $D \in M_{r\times n}(R)$. Suppose $B + A$ is a product of idempotent matrices. Then the matrix*

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$$

is also a product of idempotent matrices

Proof. Consider

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_n & -C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B + A & 0 \\ D & 0 \end{pmatrix}$$

Let us write $B + A = E_1 \cdots E_l$ where, for any $1 \le i \le n$, $E_i^2 = E_i$ we then have

$$\begin{pmatrix} B + A & 0 \\ D & 0 \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} E_l & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D & 0 \end{pmatrix}.$$

This gives the proof. □

As a consequence we have the following proposition.

Proposition 8. *Let B be a matrix in $M_n(R)$. Then the matrix*

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$$

is a product of idempotent matrices in each of the following cases:

1. *If there exists $1 \leq i \leq n$ such that $(1-e_{ii})B$ (resp. $B(I_n - e_{ii})$) is a product of idempotents.*
2. *If B is a product of elementary matrices.*
3. *If B is a permutation matrix.*
4. *if B is an upper (resp. lower) triangular matrix. In particular, if B is a diagonal matrix.*

Proof. It is enough to consider the case when $r = 1$. Indeed if $r > 1$ and we assume the result true for $r = 1$, then the matrix $A' = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(R)$ is a product of idempotent matrices. Hence $A = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}$ is a product of idempotent matrices.

(1) Let $\tilde{B} = (1 - e_{ii})B$ be the matrix obtained by replacing the i^{th} row of B by a zero row. We can then write $\tilde{B} = B - e_i B_i$, where B_i is the i^{th} row of B and e_i is the column with all entries zero except the i^{th} one which is 1. Proposition 7 then applies and gives the case when $r = 1$.

(2) It is clear that A is in fact a product of quasi elementray matrices and hence Proposition 3 shows that A is a product of idempotent matrices.

(3) Since A is clearly a quasi permutation matrix, it is a product of idempotent matrices.

(4) We consider only the case of upper triangular matrices. The proof proceeds by induction on $n \geq 1$. If $n = 1$ we have the following decomposition

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Assuming the result for upper triangular matrices of size $n - 1$, we consider the case of a matrix $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ where $B \in M_n(R)$ is upper triangular. We consider, as in (1) above, the matrix $\tilde{B} = B(I_n - e_{nn})$ obtained by replacing the last column of B by zeros. Since $\tilde{B}$ is upper triangular, the induction hypothesis shows that the matrix $\tilde{B}$ is a product of idempotent and hence (1) above shows that A is also a product of idempotent matrices. □

Reacall that R is a separative ring if for any finitely projective right modules M and N, $M \oplus N \cong N \oplus N \cong M \oplus M$ implies that $M \cong N$.

Theorem 9. *If the ring R is a von Neumann regular separative ring then for any matrix $B \in M_n(R)$ the matrix*

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R),\ r > 0$$

is a product of idempotent matrices.

Proof. If the ring R is supposed to be von Neumann regular and separative then, the Morita invariance of these properties implies that the same is true for any matrix ring over R. Let B be any matrix in $M_n(R)$. The main result of [21] shows that the matrix

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$$

is a product of idempotent matrices if and only if the following relations between the annihilators are satisfied:

$$lann(A)S = Srann(A) = S(I_{n+r} - A)S$$

where $S = M_{n+r}(R)$. Let us prove the first equality. We let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in lann(A)$$

with $X_1 \in M_n(R)$ and $X_4 \in M_r(R)$ and other matrices are of the appropriate size. We notice that if $X_1 = 0$ and $X_3 = 0$ then $X \in lann(A)$. We then get that $lann(A)$ contains the matrices with the first n columns all zero. Since the right ideal of S generated by these matrices is $S = M_{n+r}(R)$, we get that $lann(A)S = S$. Similarly we also have $Srann(A) = S$. Since $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, we can show that the ideal generated by $I_{n+r} - A$ is the ring S itself. This concludes the proof. □

Remark 10. It is an open question whether a von Neumann regular ring is always separative. Let R be a von Neumann regular ring. If there exists a matrix $B \in M_n(R)$ such that the matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+r}(R)$ cannot be written as a product of idempotent matrices then it follows from Theorem 9 that this will provide an example of a regular ring R that is not separative. If B is invertible, this would also answer the question (3) in [18].

Recall that a ring R is a GE ring, as defined by P.M. Cohn (cf. [8] p. 150), if for any $n > 0$, the group $GL_n(R)$ is generated by elementary and diagonal matrices.

It was proved in [5] that a separative exchange ring is a GE ring. In particular, any separative regular ring is GE. We now generalize Theorem 9 when B is invertible.

Theorem 11. *If the ring R is a GE ring then for any invertible matrix $B \in M_n(R)$ the matrix*

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

is a product of idempotent matrices.

Proof. If $n = 1$ we can consider the decomposition obtained in the proof of Proposition 8 (4).

So suppose that $n > 1$, and that R is a GE ring. Since B is invertible, the matrix B is a product of elementary matrices and invertible diagonal matrices. The conclusion follows easily from Proposition 8. □

We will now prove that if the matrix $A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ is of size of at least twice the size of B then A is a product of idempotent matrices.

Proposition 12. *Let $A \in M_l(R)$ be of the form*

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

where $B \in M_n(R)$ and $l \geq 2n$ then A is a product of idempotent matrices.

Proof. We first remark that, as in the begining of the proof of Proposition 8, it is enough to consider the case when $l = 2n$. So we assume that $A \in M_{2n}(R)$. Proposition 4 shows that that the given matrix is a product of idempotents if and only if the matrix

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

is a product of idempotents. But for this matrix we have the obvious decompositon

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

□

4. TOTALLY NONEGATIVE MATRICES

We conclude this paper by answering in the negative the open question 2 in [18]: is it true that a singular totally nonnegative matrix is a product of nonnegative idempotents? The following example shows that this is not true. Consider the following example of a singular totally nonnegative matrix

$$A = \begin{pmatrix} \alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ \alpha & 0 & 0 & \alpha \\ 0 & \alpha & 0 & 0 \end{pmatrix} \quad \alpha > 0$$

Let us assume that $A = E_1 \dots E_n$ where $E_i \neq I_4$ are nonnegative idempotents. Then $E_1 A = A$. Let us write

$$E_1 = \begin{pmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \\ x_4 & y_4 & z_4 & t_4 \end{pmatrix} \quad x_i, y_i, z_i, t_i > 0 \quad \text{for } 1 \leq i \leq 4$$

Comparing the first rows of E_1 and $E_1 A$, we get $x_1 + z_1 = 1,\ x_1 + t_1 = 1,\ y_1 + z_1 = 0$. This gives $x_1 = 1,\ y_1 = z_1 = t_1 = 0$. Continuing in this way with other rows leads to $E_1 = I_4$, a contradiction.

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